Moment Generating Functions (MGFs)

Definition 1.1 *The* i^{th} **moment** *of a random variable,* X*, is*

$$\mathrm{E}_X(X^i) = \mu_i' \tag{1}$$

Definition 1.2 *The* i^{th} **central moment** *of a random variable,* X*, is*

$$\mathrm{E}_X\left[(X-\mu)^i\right] = \mu_i \tag{2}$$

Note that $\mu = \mathrm{E}_X(X^1)$ is the first moment of X, and that

$$\sigma^2 = \mathrm{E}\left[(X - \mathrm{E}_X(X))^2\right] \tag{3}$$

$$= \mathrm{E}_X\left[(X-\mu)^2\right] \tag{4}$$

is the second central moment of X.

Further, $\sigma^2 = \mathrm{E}_X(X^2) - [\mathrm{E}_X(X)]^2$ is expressible as a function of noncentral moments.

Definition 1.3 *Suppose that* X *is a random variable with a given pmf, and that, for some real* $h > 0$, $\mathrm{E}_X(e^{sX})$ *exists for every value of* $s \in (-h, h)$*. Then the function* M *defined by*

$$M(s) = \mathrm{E}_X(e^{sX}) \tag{5}$$

is called the **moment generating function** *of* X*.*

Hence,

$$M(s) = \begin{cases} \sum_x e^{sx} p_X(x) & \text{discrete} \\ \int_{-\infty}^{\infty} e^{sx} f_X(x) dx & \end{cases} \tag{6}$$

Why do we care? Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{7}$$

Hence,

$$e^{sX} = \sum_{r=0}^{\infty} \frac{(sX)^r}{r!} \tag{8}$$

and thus,

$$M(s) = \mathrm{E}_X\left[\sum_{r=0}^{\infty} \frac{(sX)^r}{r!}\right] \tag{9}$$

$$= \sum_{r=0}^{\infty} \mathrm{E}_X(X^r) \frac{s^r}{r!} \tag{10}$$

We now note that the coefficient of $\frac{s^r}{r!}$ is $\mathrm{E}_X(X^r)$.

Now, we also note that, by Maclaurin series expansion,

$$M(s) = \sum_{r=0}^{\infty} \frac{d^r}{ds^r} M(s)\bigg|_{s=0} \frac{s^r}{r!} \tag{11}$$

$$= \sum_{r=0}^{\infty} M^{(r)}(0) \cdot \frac{s^r}{r!} \tag{12}$$

Comparing the coefficients of $\frac{s^r}{r!}$ we see that

$$\mathrm{E}_X(X^r) = M^{(r)}(0) \tag{13}$$

$$= \frac{d^r}{ds^r} M(s)\bigg|_{s=0} \tag{14}$$

Hence, given an mgf we may generate moments until we are blue in the face.

Example 1.1 *Consider $X \sim$ Bernoulli(p).*

$$M(s) = e^{s(0)} p_X(0) + e^{s(1)} p_X(1) \tag{15}$$

$$= 1(1-p) + e^s(p) \tag{16}$$

$$= pe^s + 1 - p \qquad -\infty < s < \infty \tag{17}$$

Thus,

$$\mathrm{E}_X(X) = \frac{d}{ds} M(s)\bigg|_{s=0} \tag{18}$$

$$= pe^s|_{s=0} \tag{19}$$

$$= p \tag{20}$$

$$\mathrm{E}_X(X^2) = \frac{d^2}{ds^2} M(s)\bigg|_{s=0} \tag{21}$$

$$= pe^s|_{s=0} \tag{22}$$

$$= p \tag{23}$$

$$\mathrm{E}_X(X^3) = \frac{d^3}{ds^3} M(s)\bigg|_{s=0} \tag{24}$$

$$= pe^s|_{s=0} \tag{25}$$

$$= p \tag{26}$$

This simplifies the computation of

$$\mathrm{Var}(X) = p - p^2 \tag{27}$$

$$= p(1-p) \tag{28}$$

Example 1.2 *Suppose $X \sim N(0,1)$. Then*

$$
\begin{aligned}
M(s) &= \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx &(29)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2-2sx)/2} dx &(30)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left[-(x^2-2sx+s^2)/2\right]+s^2/2} dx &(31)\\
&= e^{s^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-s)^2/2} dx &(32)\\
&= e^{s^2/2} \quad -\infty < s < \infty &(33)
\end{aligned}
$$

1.1 Characteristics of MGFs

Theorem 1.1 *Suppose a random variable, X, has the mgf M_X. Let $Y = aX + b$, for $a \in \Re$ and $b \in \Re$. Then the mgf of Y is*

$$
\begin{aligned}
M_Y(s) &= M_{aX+b}(s) &(34)\\
&= e^{bs} M_X(as) &(35)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
M_{aX+b}(s) &= \mathrm{E}_X\left(e^{(aX+b)s}\right) &(36)\\
&= \mathrm{E}_X\left(e^{bs} \cdot e^{asX}\right) &(37)\\
&= e^{bs}\mathrm{E}_X\left(e^{asX}\right) &(38)\\
&= e^{bs} M_X(as) &(39)
\end{aligned}
$$

Example 1.3 *Consider $Y \sim N(\mu, \sigma^2)$. Find the mgf of Y.*

Solution: Let $Z = \frac{Y-\mu}{\sigma}$. Then $Z \sim N(0,1)$. Now, $Y = \mu + \sigma X$, and thus

$$
\begin{aligned}
M_Y(s) &= M_{\mu+\sigma X}(s) &(40)\\
&= e^{\mu s} M_X(\sigma s) &(41)
\end{aligned}
$$

But, since $Z \sim N(0,1)$, $M_X(s) = e^{s^2/2}$ and

$$
\begin{aligned}
M_Y(s) &= e^{\mu s} e^{(\sigma^2 s^2/2)} &(42)\\
&= e^{\mu s + (\sigma^2 s^2/2)} \quad -\infty < s < \infty &(43)
\end{aligned}
$$

Theorem 1.2 *If X and Y are independent random variables with mgf's $M_X(s)$ and $M_Y(s)$ then for $a, b \in \Re$*

$$M_{aX+bY}(s) = M_X(as) \cdot M_Y(bs) \qquad (44)$$

Example 1.4 *Recall that if* $X \sim B(n,p)$ *then* $X = Y_1 + Y_2 + \cdots + Y_n$ *where* $Y_i \overset{iid}{\sim}$ *Bernoulli*(p). *Since* $M_{Y_i}(s) = pe^s + (1-p)$ *for* $i = 1, /2, \ldots,\ n,$

$$M_X(s) = M_{Y_1}(s) M_{Y_2}(s) \cdots M_{Y_n}(s) \quad (45)$$

$$= [pe^s + (1-p)]^n \quad (46)$$

This points toward a general result which is of major importance.

Theorem 1.3 *Two random variables share a common mgf if and only if they share a common distribution.*

I.e. moment generating functions are unique.